A TWO PHASE SEQUENTIAL PROBABILITY RATIO TEST

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ABSTRACT

For the problem if discriminating between two simple hypotheses about a normal mean, a sequential test procedure carried out in two phases is proposed. Read's partial sequential probability ratio test can be studied as a special case of the proposed procedure.

INTRODUCTION

Wald's (1947) sequential probability ratio test (SPRT) for testing a simple hypothesis $H_0 : \mu = \mu_0$ against a simple alternative $H_1 : \mu = \mu_1, \mu_1 > \mu_0$ about a parameter μ has a drawback in that if μ is between μ_0 and μ_1, the average sample number (ASN) may even be higher than the sample-size of a fixed-sample test having the same error probabilities. To overcome this defect, Read (1971) introduced the partial sequential probability ratio test (PSPRT) in which an initial fixed number n of observations is followed by Wald's SPRT procedure. Read's computations show that at least for the problem of testing a normal mean with known variance, the ASN of the PSPRT at $\mu = \mu^* = (\mu_o + \mu_1)/2$ is substantially less than the corresponding ASN of the SPRT for preassigned error probabilities.

Here is an attempt to generalize Read's idea. Instead of drawing a fixed sample prior to the Wald's SPRT, we draw observations sequentially with diverging boundaries, and this is the first phase of the proposed two-phase procedure. After n observations, if the procedure is not terminated till then, Wald's SPRT is started in the second phase with upper and lower boundaries equal respectively to the upper and the lower boundary points in the starting of the test procedure. Also, at the nth stage of sampling if the sample-path stays below the Wald's line of acceptance or above the Wald's line of rejection, decisions are made accordingly. For a finite n, if the boundaries in the first phase diverge to infinity, the proposed test procedure takes the form of a PSPRT.

Although the procedure will be discussed as a generalization of the PSPRT, the aim and objective of the discussion is no more than establishing that if the Wald-boundaries are broken at some point of the sample number axis and if prior to that some other continuation region with either converging or diverging lines is used, the maximum ASN can be lowered substantially, and in fact, only a special case of this phenomenon was established by Read (1971) in his PSPRT.

1. The Procedure

Let $X_1, X_2, \ldots$ be a sequence of independently and identically distributed random variables following the normal law with unknown mean μ and variance unity. Consider the problem of testing $H_o: \mu = \mu_o$ versus $H_1: \mu = \mu_1, \mu_1 > \mu_o$.

Replacement of the observations x by $(x - \mu^*)$ gives

$$\log (p_{ij}/p_{oj}) = \Delta y_j \tag{1.1}$$

where $\Delta = \mu_1 - \mu_o$, p_{ij} is the joint likelihood of the first j-observations prior to the aforesaid transformation under H_i, i = 1, 2 and $y_j = \sum_{i=1}^{j} x_i$.

Now, Wald's SPRT consists in drawing observations sequentially according to whether

$$c_2 < y_j < c_1 \tag{1.2}$$

or not, for any j and for $c_2 < 0 < c_1$. If at any stage-j , (1.2) is violated on either side, the sampling is terminated, H_1 is accepted if $y_j \geqslant c_1$ while H_o is accepted if $y_j \leqslant c_2$.

The proposed two-phase SPRT T_n, θ_1, θ_2 is defined as follows: Given an integer n , angles θ_1, θ_2, ($0^\circ < \theta_i < 90^\circ$, i = 1, 2) and boundaries c_1 and c_2, $c_2 < 0 < c_1$, observations are drawn according to whether

$$a_2(j) < y_j < a_1(j) \tag{1.3}$$

or not, where

$$a_i(j) = \begin{cases} c_i + (-1)^{i+1} j \tan \theta_i, & i = 1, 2, j < n \\ e_i & i = 1, 2, j \geqslant n \end{cases} \tag{1.4}$$

If at any stage (1.3) is violated the experimentation stops with acceptance of H_o if the left inequality is violated, otherwise with acceptance of H_1 if the right inequality is violated.

For θ_1 and θ_2 equal to 0°, T_n, θ_1, θ_2 is nothing but Wald's SPRT and for θ_1 and $\theta_2 \to 90^\circ$ what we get is Read's PSPRT.

To calculate the operating characteristic (OC) and the ASN functions, we replace y_j, j = 1, 2, ... by an analogous x(t), $0 < t < \infty$, t being in the continuous sense (cf. Anderson (1960)). X(t) is a Weiner stochastic process with mean μt and variance t. In the first phase of sampling the process X(t) has the continuation region bounded by the upper line $y = c_1 + d_1 t$, and the lower line $y = c_2 + d_2 t$, in the (t, X(t)) plane, where

$$\left.\begin{aligned} d_1 &= \tan\theta_1 \\ d_2 &= -\tan\theta_2 \end{aligned}\right\} \tag{1.5}$$

2. The OC Function

Let $L(\mu)$ be the probability of accepting H_0 at the parameter point μ. We can write

$$L(\mu) = L_1(\mu) + L_2(\mu) + L_3(\mu) \tag{2.1}$$

where $L_1(\mu)$ is the probability that $X(t) \leqslant c_2 + d_2 t$ for some $t \leqslant n$ which is smaller than any t for which $X(t) \geqslant c_1 + d_1 t$; $L_2(\mu)$ is the probability that $X(n) \in [c_2 + d_2 n, c_2]$ given that $X(t) \in (c_2 + d_2 t, c_1 + d_1 t)$ for all $t < n$; $L_3(\mu)$ is the probability that $X(\zeta) \leqslant c_2$ for some $\zeta > n$ given that $X(t) \in (c_2 + d_2 t, c_1 + d_1 t)$ for all $t \leqslant n$.

$L_1(\mu)$ can be obtained from Anderson's (1960) equation (5.5), which after interchanging (γ_1, δ_1) with $(-\gamma_2, -\delta_2)$ where $\gamma_i = c_i$, $\delta_i = d_i - \mu$, $i = 1,2$ gives

$$\begin{aligned} L_1(\mu) = \sum_{r=0}^{\infty} \{ & e^{-2[r\gamma_1 - (r+1)\gamma_2][r\delta_1 - (r+1)\delta_2]} \\ & \cdot \Phi\left(\frac{-\delta_2 n - 2r\gamma_1 + (2r+1)\gamma_2}{\sqrt{n}}\right) \\ + & e^{-2[r^2\gamma_1\delta_1 + r^2\gamma_2\delta_2 - r(r+1)\gamma_2\delta_1 - r(r-1)\gamma_1\delta_2]} \\ & \cdot \Phi\left(\frac{\delta_2 n - 2r\gamma_1 + (2r+1)\gamma_2}{\sqrt{n}}\right) \\ - & e^{-2[(r+1)^2\gamma_1\delta_1 + (r+1)^2\gamma_2\delta_2 - r(r+1)\gamma_2\delta_1 - (r+1)(r+2)\gamma_1\delta_2]} \\ & \cdot \Phi\left(\frac{-\delta_2 n - 2(r+1)\gamma_1 + (2r+1)\gamma_2}{\sqrt{n}}\right) \end{aligned}$$

$$- e^{-2\,[(r+1)\gamma_1 - r\gamma_2]\,[(r+1)\delta_1 - r\delta_2]} \cdot \Phi\left(\frac{\delta_1 n - 2(r+1)\gamma_1 + (2r+1)\gamma_2}{\sqrt{n}}\right) \tag{2.2}$$

where Φ (x) is the unit normal c.d.f.

Again, Anderson's (1960) theorem (4.2) gives the probability $P_1(n, x)$ of crossing the upper line first before crossing the lower line up to and including the first n observations given that $X(n) = x \leqslant c_1 + d_1 n$. Similaly, it gives the probability $P_2(n, x)$ of crossing the lower line first before crossing the upper line up to and including the first n observations given that $X(n) = x \geqslant c_2 + d_2 n$. Clearly,

$$P(n, x) = 1 - P_1(n, x) - P_2(n, x), h_2 < c_2 \leqslant x \leqslant c_1 < h_1 , \tag{2.3}$$

gives the probability that the process crosses neither the upper line nor the lower line up to and including the first n observations, where $h_i = c_i + d_i n$, i = 1, 2. So we get

$$L_2(\mu) = \int_{h_2}^{c_2} p(n, x) \frac{1}{\sqrt{2\pi}\sqrt{n}} e^{\frac{-(x - n\mu)^2}{2n}} dx \quad 2.4)$$

where

$$P(n, x) = 1 - \sum_{i=1}^{2} \sum_{r=1}^{\infty} \sum_{j=1}^{2} (-1)^{j+1} e^{v_{ij} + x u_{ij}} ,$$

$$u_{ij} = -2\,\{r(c_i' - c_i) - (2 - j)\; c_i'\}/n,$$

$$v_{i1} = -2\;[r^2\, c_i h_i + (r-1)^2\; c_{i'} h_{i'} - r(r-1) \cdot (c_1 h_2 + c_2 h_1)]/n, \tag{2.5}$$

$$v_{i2} = -2\ [r^2(c_1h_1 + c_2h_2) - r(r-1)\, c_ih_i{}' - r(r+1)$$

$$\cdot\ c_i{}'h_i]/n\ ,$$

for $\qquad i' = 1 + i \pmod 2$

which ultimately leads us to

$$L_2\,(\mu) = \Phi\,(\frac{c_2 - n\mu}{\sqrt{n}}) - \Phi\,(\frac{h_2 - n\mu}{\sqrt{n}})$$

$$- \sum_{i=1}^{2} \sum_{r=1}^{\infty} \sum_{j=1}^{2} (-1)^{j+1} \quad e^{v_{ij} + (2n\mu\mu_{ij} + n\mu_{ij}{}^2)/2}$$

$$\cdot \{\Phi\,(\frac{c_2 - n\mu - nu_{ij}}{\sqrt{n}}) - \Phi\,(\frac{h_2 - n\mu - nu_{ij}}{\sqrt{n}})\} \qquad (2.6)$$

Next, let (P(x) be the probability of accepting the null hypothesis given that the procedure initially starts at some point x. For the problem of testing a normal mean, (cf. Billard (1973), Read (1971)) Wald's (1974) expression for the OC gives us

$$P(x) = \frac{\exp(-2\mu c_1) - \exp(-2\mu n)}{\exp(-2\mu c_1) - \exp(-2\mu\, c_2)}\ , \quad \mu \neq \mu^* \qquad (2.7)$$

where c_1, c_2 $(c_2 < 0 < c_1)$ are Wald's boundaries.

As it is customary to have sampling plans with

Prob. (H_o will be rejected) $\leqslant \alpha$ for $\mu = \mu_o$

and (2.8)

Prob. (H_o will be accepted) $\leqslant \beta$ for $\mu = \mu_1$

for given errors α and β, we shall be concerned with examining T_n, ϕ_1, ϕ_2 with respect to the ASN's at μ_o, μ^* and μ_1 for pre-assigned errors and hence we exclude the case $\mu = \mu^*$ while discussing about the OC function, although it can be very easily found out using L'Hospital's rule.

Clearly,

$$L_3(\mu) = \int_{c_2}^{c_1} P(n, x) \cdot p(x) \cdot \frac{1}{\sqrt{2\pi}\sqrt{n}} e^{-\frac{(x - n\mu)^2}{2n}} dx \tag{2.9}$$

Using

$$\Delta_i = (c_i - n\mu)/\sqrt{n}, \quad i = 1, 2$$

and

$$f(t) = \exp\{(nt^2 + 2nt\mu)/2\}$$

(2.9) can be simplified to

$$\begin{aligned}(e^{-2\mu c_1} - e^{-2\mu c_2}) \cdot L_3(\mu) &= \epsilon^{-2\mu c_1} \{\Phi(\Delta_1) - \\ &\Phi(\Delta_2)\} - \{\Phi(\Delta_1 + 2\mu\sqrt{n}) - \Phi(\Delta_2 + 2\mu\sqrt{n})\} \\ &- \sum_{i=1}^{2} \sum_{r=1}^{\infty} \sum_{j=1}^{2} (-1)^{j+1} \cdot \{ e^{v_{ij} - 2\mu c_1} f(u_{ij}) \cdot \\ &[\Phi(\Delta_1 - u_{ij}\sqrt{n}) - \Phi(\Delta_2 - u_{ij}\sqrt{n})] \\ &- e^{v_{ij}} \cdot f(u_{ij} - 2\mu) \cdot [\Phi(\Delta_1 - (u_{ij} - 2\mu)\sqrt{n}) \\ &- \Phi(\Delta_2 - (u_{ij} - 2\mu)\sqrt{n})]\}\end{aligned} \tag{2.10}$$

Finally, the OC at μ can be obtained using (2.1). It can be seen that when θ_1 and $\theta_1 \rightarrow 90^\circ$ in the limit, (2.1) reduces to Read's (1971) equation (3.5).

3. The ASN Function

With m as the decisive sample number (DSN) of T_n, θ_1, θ_2 and q(m) as the probability mass function of m, the ASN will be given by

$$E(\mu) = \sum_{m=1}^{n} mq(m) + \sum_{m'=1}^{\infty} m'q(m'+n) + n \sum_{m'=1}^{\infty} q(m'+n) \tag{3.1}$$

where $m' = m - n$. Let

$$\sum_{m=1}^{n} mq(m) = E_1(\mu) \tag{3.2}$$

so that $E_1(\mu)$ is the average of the DSN up to and including the first n observations. Hence,

$$\sum_{m=1}^{n} mq(m) + n \sum_{m'=1}^{\infty} q(m'+n) = E_1(\mu) + n \cdot \text{Prob}\,(m \geqslant n) \tag{3.3}$$

together constitute the ASN function of an Anderson-type procedure with diverging boundaries. In Anderson's notations

$$E_i(\mu) = \epsilon_1^* + \epsilon_2^* \tag{3.4}$$

where ϵ_1^* is the contribution to the ASN in the sense that the upper line was crossed first and $\epsilon_2{}^*$ is the contribution to the ASN in the sense that the lower line was crossed first, that is,

$$\epsilon_1{}^* = \frac{1}{\delta_1} \sum_{r=0}^{\infty} \{[e^{-2\,[(r+1)\gamma_1 - r\gamma_2]\,[(r+1)\delta_1 - r\delta_2]} \cdot \Phi \left(\frac{\delta_1 n + 2r\gamma_2 - (2r+1)\gamma_1}{\sqrt{n}}\right)$$

$$- e^{-2\,[r^2\gamma_1\delta_1 + r^2\gamma_2\delta_2 - r(r+1)\gamma_1\delta_2 - r(r-1)\gamma_2\delta_1]} \cdot \Phi \left(\frac{-\delta_1 n + 2r\gamma_2 - (2r+1)\gamma_1}{\sqrt{n}}\right)]$$

$$[(2r+1)_1 - 2r\gamma_2] - [e^{-2\,[(r+1)^2\gamma_1\delta_1 + (r+1)^2\gamma_2\delta_2 - r(r+1)\gamma_1\delta_2 - (r+1)(r+2)\gamma_2\delta_1]} \cdot$$

$$\Phi \left(\frac{\delta_1 n + 2(r+1)\gamma_2 - (2r+1)\gamma_1}{\sqrt{n}}\right)$$

$$- e^{-2\,[r\gamma_1 - (r+1)\gamma_2] \cdot [r\delta_1 - (r+1)\delta_2]}$$

$$\cdot \Phi \left(\frac{-\delta_1 n + 2(r+1)\gamma_2 - (2r+1)\gamma_1}{\sqrt{n}}\right)] \cdot [(2r+1)\gamma_1$$

$$- 2(r+1)\gamma_2]\}, \delta_1 \neq 0 \cdot \qquad (3.5)$$

Interchanging (γ_1, δ_1) with $(-\gamma_2 - \delta_2)$ in (3.5) ϵ_2^* can be obtained.

In Section 2, $L_1(\mu)$ was defined as the probability that $X(t) \leqslant c_2 + d_2 t$ for some $t \leqslant n$ which is smaller than any t for which $X(t) \geqslant c_1 + d_1 r$. In the expression of $L_1(\mu)$ (equation 2.2) interchanging (γ_1, δ_1) with $(-\gamma_2, -\delta_2)$ we obtain the probability $L_1'(\mu)$ that $X(t) \geqslant c_1 + d_1 t$ for some $t \leqslant n$ which is smaller than any t for which $X(t) \leqslant c_2 + d_2 t$. It can be seen that (cf. Anderson (1960))

$$1 - L_1(\mu) - L_1'(\mu) = \text{Prob}(m \geqslant n) \tag{3.6}$$

The second term in (3.1) is the average of the additional observations required for a decision after n observations, given that the experimentation does not terminate prior to that. With n(x) as the average number of observations required for termination of sampling given that the procedure starts initially at some point x, we obtain

$$\sum_{m'=1}^{\infty} m' q(m' + n) = \int_{c_2}^{c_1} P(n, x) \cdot n(x) \cdot \frac{1}{\sqrt{2\pi}\sqrt{n}} \cdot e^{-\frac{(x - n\mu)^2}{2r}} dx = E_2(\mu), \quad \text{say} \tag{3.7}$$

For the problem of testing a normal mean (cf. Billard (1973), Read (1971)) Wald's (1947) expressions for the ASN give

$$n(x) = \begin{cases} = \dfrac{1}{\mu} \left\{ \dfrac{(c_1 - c_2)\exp(-2\mu x) - c_1 \exp(-2\mu c_2) + c_2 \exp(-2\mu c_1)}{\exp(-2\mu c_1) - \exp(-2\mu c_2)} \right. \\ \left. - x \right\}, \ \mu \neq \mu^* \\ = (c_1 - x)(x - c_2), \ \mu \neq \mu^*; \end{cases} \tag{3.8}$$

This gives,

$$E_2(\mu) = r_1 G_2 - r_2 G_1 + \frac{\sqrt{n}}{\mu} \cdot g$$

$$- \sum_{i=1}^{2} \sum_{r=1}^{\infty} \sum_{j=1}^{2} (-1)^{j+1} \cdot \{ r_1 (G_2 e^{v_{ij}} + G_2' \cdot f(u_{ij} - 2\mu))$$

$$- r_2 (G_1 \cdot e^{v_{ij}} + G_1' \cdot f(u_{ij}))$$

$$+ \frac{\sqrt{n}}{\mu \sqrt{2\pi}} (e^{z_{ij1}} - e^{z_{ij2}}) + \frac{\sqrt{n}}{\mu} \cdot f(u_{ij}) \cdot g' \} ,$$

$$\text{for } \mu \neq \mu^* \qquad (3.9a)$$

and

$$E_2(\mu) = (-n - c_1 c_2) \cdot \{ \Phi (c_1/\sqrt{n}) - \Phi (c_2/\sqrt{n}) \}$$

$$- c_2 \sqrt{n} \; \Phi (c_1/\sqrt{n}) + c_1 \sqrt{n} \; \Phi (c_2/\sqrt{n})$$

$$- \sum_{i=1}^{2} \sum_{r=1}^{\infty} \sum_{j=1}^{2} (-1)^{j+1} \{ e^{(v_{ij} + n u_{ij}^2)/2}$$

$$\cdot [\Phi (f_1) - \Phi (f_2)] \cdot$$

$$\cdot [n u_{ij} (c_1 + c_2) - n - n^2 u_{ij}^2 - c_1 c_2]$$

$$- [\Phi (f_1) - \Phi (f_2)] \cdot [\sqrt{n} ((c_1 + c_2) - 2 n u_{ij})]$$

$$+ f_1 \cdot n \; \Phi (f_1) - f_2 \cdot n \; \Phi (f_2) \} , \text{ for } \mu = \mu^* \qquad (3.9b)$$

where

$$r_1 = (c_1 - c_2)/(\mu x_2) ,$$

$$r_2 = n + x_1/(\mu x_2) ,$$

$$\Delta_x' = \Delta_x - u_{ij}\sqrt{n}\ , \ x = 1, 2$$

$$Z_{ijx} = v_{ij} - \Delta_x^2/2, \ x = 1, 2 ,$$

$$X_1 = c_1 \exp(-2\mu c_2) - c_2 \exp(-2\mu c_1) ,$$

$$X_2 = \exp(-2\mu c_1) - \exp(-2\mu c_2) ,$$

$$G_1 = \Phi(\Delta_1) - \Phi(\Delta_2) , \ G_1' = \Phi(\Delta_1') - \Phi(\Delta_2') ,$$

$$G_2 = \Phi(\Delta_1 + 2\mu\sqrt{n}) - \Phi(\Delta_2 + 2\mu\sqrt{n}) ,$$

$$G_2' = \Phi(\Delta_1' + 2\mu\sqrt{n}) - \Phi(\Delta_2' + 2\mu\sqrt{n}) ,$$

$$g = \Phi(\Delta_1) - \Phi(\Delta_2) ,$$

$$g' = \Phi(\Delta_1') - \Phi(\Delta_2') ,$$

$$f_x = (c_x - nu_{ij})/\sqrt{n}\ , \ x = 1, 2,$$

and $$\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Finally, (3.1) gives us the ASN. As θ_1 and θ_2 tend to 90° in the limit, (3.1) leads us to Read's (1971) equations (4.1) and (4.3) according as $\mu \neq \mu^*$ or $\mu = \mu^*$.

4. NUMERICAL RESULTS AND DISCUSSIONS

Read (1971) has shown that at least for the case of testing a normal mean, if an initial fixed-sample is followed by an Wald's SPRT, the ASN at μ^* can be considerably reduced. Our aim was to examine the effect on the ASN at at μ^* by replacing the fixed-sample in the Read's PSPRT with a sequential plan with diverging boundaries.

Preliminary computations show that the OC at μ_o is reduced when the Wald-lines are replaced by certain T_n, θ_1, θ_2 boundaries. To make a meaningful comparison of the ASN's of the present procedure with those of other existing procedures, the first and the second kinds of error of the respective procedures should be same. For $\Delta = 0.20$, n = 100 (20) 200 and for integral values of θ $(= \theta_1 = \theta_2)$ the equation

$$L(\mu_o) = 0.95 \tag{4.1}$$

was solved by Regula-Falsi for c $(= c_1 = -c_2)$ so that the nominal errors α' and β' are equal to 0.05. Some of the computed values are presented in table-1 below.

From table-1 we may note that with the same n as the optimal PSPRT both at μ^* and at μ_o (or μ_1) the ASN's could be made lower by using the proposed two phase procedure. Also, these computed values establish that the two-phase procedure has the property of reducing the ASN at μ^* considerably; it can be observed that for certain combinations of the parameters θ and n, the ASN's at μ^* as well as at μ_o are smaller than the respective values attainable by even the optimal PSPRT procedure. Again a detailed analysis of Anderson's computed values (Table-1 of Anderson's (1960) paper) reveals that the probabilities of continuation of the Anderson's procedure with converging lines after the points of truncation are so small that further contributions towards the ASN's will be negligible even if the procedures are continued using Wald-type boundaries after the points of truncation. Thus it can be stated that reduction of the maximum ASN in a two-phase SPRT with either converging or diverging boundaries in the first phase followed by Wald's SPRT in the second phase, is an inherent feature of the test procedure and actually only a special case of this phenomenon was established in Read's PSPRT.

Further computations with different ranges of n for different integral values of θ show that the minimum ASN at μ^* occurs at 4 degrees after which the ASN at μ^* shows slow increase to Read's value as θ increases. And, as expected, the ASN at μ_o (or μ_1) increases from Wald's value as θ increases. Table-2 below shows the minimum attainable ASN at μ^* for various θ values and the corresponding ASN's at μ_o.

Table 1

Values of ASN at μ^* (Top) and at μ_0 (Bottom) and
Corresponding values of c (within parentheses) for n = (100(20)160,
$\theta = 3^\circ (1^\circ) 6^\circ$ and 90°, $\Delta = 0.20(**)$, $\alpha' = \beta' = 0.05$

n \ θ°	*3*	*4*	*5*	*6*	*90(*)*
100	207.865, 136.551, (13.769)	207.111, 137.402, (13.625)	206.836, 138.478, (13.529)	206.819, 139.608, (13.465)	207.764, 145.202, (13.353)
120	205.748, 138.180, (13.357)	204.905, 139.692, (13.130)	204.681, 141.556, (12.971)	204.790, 143.525, (12.861)	206.736, 153.890, (12.643)
140	204.536, 139.679, (12.940)	203.762, 141.931, (12.620)	203.725, 144.697, (12.384)	204.087, 147.661, (12.212)	207.551, 167.674, (11.819)
160	204.328, 140.984, (12.541)	203.809, 143.984, (12.123)	204.102, 147.673, (11.801)	204.829, 151.684, (11.556)	210.507, 177.310, (10.884)

(* values computed using Read's expressions directly.)
(** Read's (1971) table-2 appears to have a printing error.
The computed values in that table are for Δ= 0.20 and not for Δ= 0.25 as stated.)

Table 2

Minimum Attainable ASN's at μ^*, corresponding ASN's at μ_o (or μ_1) and the appropriate combinations of c and n For $\theta = 0^o(1^o)6^o, \Delta = 0.20$ and $\alpha' = \beta' = 0.05$

θ (in degrees)	c	n	ASN(μ^*)	(ASN(μ_o)
0 (*)	14.720	∞	216.70	132.50
1	13.754	165.0	210.01	139.06
2	13.097	159.0	206.15	139.08
3	12.657	154.0	204.27	140.61
4	12.368	150.0	203.62	142.98
5	12.266	144.0	203.68	145.31
6	12.212	140.0	204.09	147.66

(*values computed using Wald's expressions)

The minimum attainable ASN at μ^* should therefore be available around θ = 4 degrees. But further computations reveal that the change is negligible.

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